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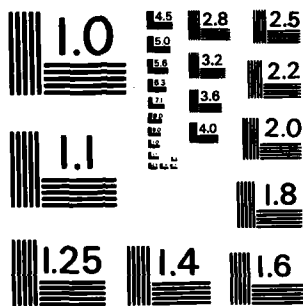
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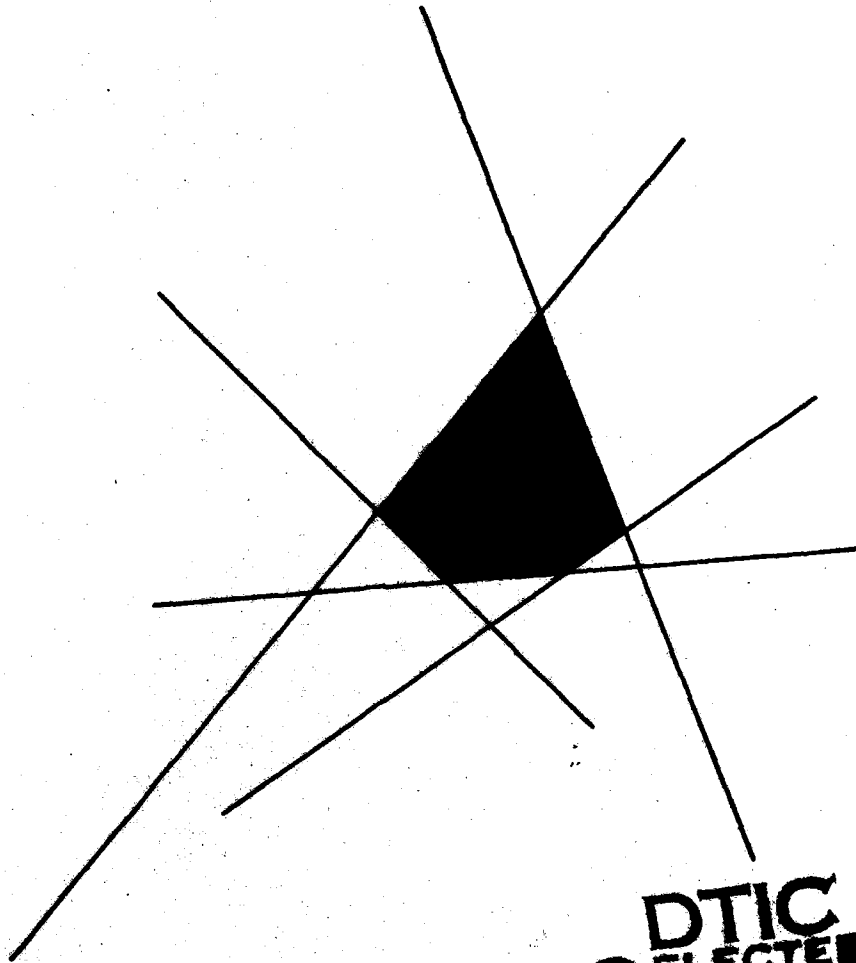
# A MODEL IN WHICH COMPONENT FAILURE RATES DEPEND ON THE WORKING SET

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SHELDON M. ROSS

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ABSTRACT

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We consider a multicomponent system in which the failure rate of a given component at any time depends on the set of working components at that time. Sufficient conditions are presented under which such a system has a new better then used life distribution. When the failed components are allowed to be repaired, we present conditions under which the resulting process is time reversible.

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# A MODEL IN WHICH COMPONENT FAILURE RATES DEPEND ON THE WORKING SET

by

Sheldon M. Ross

## 1. INTRODUCTION

Consider an  $n$  component system having some arbitrary monotone coherent structure (see Barlow and Proschan [1] for suitable definitions). We suppose that each component is initially on and stays on for a random time at which it fails. The problem of interest is to characterize the distribution of the time until the system fails. Whereas this problem is usually considered under the assumption that the component lives are independent, we are concerned with the following model which allows for dependencies in these life distributions: We suppose a Markovian model in which the failure rate of a given component at any time is allowed to depend on the set of working components at that time. Specifically, we suppose that if at some time  $W$ ,  $W \subset \{1, 2, \dots, n\}$ , represents the set of working components then for  $i \in W$  the instantaneous failure rate for component  $i$  is  $\lambda_i(W)$ .

We start by giving a sufficient condition for the distribution of system life to be NBU where we say that the nonnegative random variable  $T$  has a NBU (new better than used) distribution if

$$P\{T > s + t \mid T > s\} \leq P\{T > t\} \quad \text{for all } s, t \geq 0.$$

In words, the above states that the probability a used item survives an additional  $t$  time units is less than the corresponding probability of a new item. We are now ready to show that if the failure rate of a component increases as the set of working components decrease then system life is NBU.

Proposition 1:

If for all sets  $W_1 \subseteq W_2$ ,

$$\lambda_i(W_1) \geq \lambda_i(W_2) \quad , \quad i \in W_1 ,$$

then the time until system failure is NBU.

Proof:

For any set of components  $W$ , let  $T_W$  denote the time until the system fails when  $W$  consists of the set of components that are initially working. We will start by showing that if  $Z \subseteq W$  then  $T_Z \leq_{st} T_W$ . The proof of this will be by induction on  $k = |Z| + |W|$ , where  $|U|$  equals the number of elements in  $U$ . It is obvious for  $k = 1$  (for in this case  $Z = \phi$  and so  $T_Z \equiv 0$ ), and so assume it whenever  $|Z| + |W| = k$ . Now suppose that  $Z \subseteq W$  and  $|Z| + |W| = k + 1$ . For  $i \in Z$ , define  $X_i$  to be an exponential random variable with rate  $\lambda_i(Z)$ . Also for  $j \in W - Z$  define  $Y_j$  to be exponential with rate  $\lambda_j(W)$ . In addition, suppose that all the  $X_i$  and  $Y_j$  so defined are independent. Now let

$$X = \min \left\{ \min_{i \in Z} X_i , \min_{j \in W-Z} Y_j \right\} .$$

There are two cases we need consider:

Case 1:  $X = X_i$  for  $i \in Z$

In this case, we can set



$$T_Z = X + T_{\{Z-1\}}^*$$

$$T_W = \begin{cases} X + T_{\{W-1\}}^* & \text{with probability } \frac{\lambda_1(W)}{\lambda_1(Z)} \\ X + T_W^* & \text{with probability } 1 - \frac{\lambda_1(W)}{\lambda_1(Z)} \end{cases}$$

where  $T_U^*$  is meant to be a random variable independent of all the  $X_i$  and  $Y_j$  and with the same distribution of  $T_U$ . Since  $\{Z-1\} \subseteq \{W-1\} \subseteq W$ , it follows by the induction hypothesis that in this case  $T_Z \underset{\text{st}}{\leq} T_W$ .

Case 2:  $X = Y_j$  for some  $j \in W - Z$

In this case, we set

$$T_Z = X + T_Z^*$$

$$T_W = X + T_{\{W-j\}}^* .$$

As  $Z \subseteq \{W-j\}$ , it again follows by the induction hypothesis that

$$T_Z \underset{\text{st}}{\leq} T_W .$$

Hence, for  $Z \subseteq W$ ,  $T_Z \underset{\text{st}}{\leq} T_W$ . Now suppose all components are initially on and that the system is still working at time  $s$ . Now no matter what the set of working components is at time  $s$ , it follows from the above that the remaining life is stochastically smaller than  $T_{\{1,2,\dots,n\}}$  which proves the proposition. ||

Remarks:

- (i) Proposition 1 need not be true without the monotonicity assumption on  $\lambda_i(W)$ . For a counterexample, consider a parallel system with  $n = 2$  and

$$\lambda_1(1,2) = \lambda_2(1,2) = 1$$

$$\lambda_1(1) = \epsilon, \quad \lambda_2(1) = 1.$$

Suppose the system is working at  $t$ . Now as  $t$  becomes larger at some point the system's failure rate starts to decrease because it becomes more and more likely that only component 1 is working (the only other possibility of any probability being that only 2 is working). Hence, system life will not be NBU.

- (ii) As the failure rate of a working component depends on the set of failed components, the question arises as to whether Proposition 1 would remain true if this failure rate were allowed to depend on the order in which the components have failed. That is, suppose that  $\bar{\lambda}_i(i_1, i_2, \dots, i_k)$ ,  $i \neq i_j$ ,  $j = 1, \dots, k$ , is the failure rate of component  $i$  when components  $i_1, \dots, i_k$  have failed and in that order. Would system life be NBU if  $\bar{\lambda}_i(i_1, \dots, i_k) \leq \bar{\lambda}_i(i_1, \dots, i_k, i_{k+1})$ ? The answer is no for consider the following example for a parallel system:

$$n = 3, \quad \bar{\lambda}_1 = \bar{\lambda}_2 = 10^6, \quad \bar{\lambda}_3 = 1$$

$$\bar{\lambda}_3(1,2) = 10, \quad \bar{\lambda}_3(2,1) = 1$$

where  $\bar{\lambda}_1$  is the initial failure rate of component 1. Now after a short time both 1 and 2 would be failed and so the remaining life will be a mixture of an exponential with rate 10 (if 1 failed before 2) or an exponential with rate 1 (if 2 failed before 1). But a mixture of exponentials with unequal rates has a decreasing failure rate and so system life could not be NBU.

(iii) When  $n = 2$ , the joint distribution of the lifetimes of the two components is called the Freund Distribution (see [2]). It can be shown in this case (see [6]) that the time of system failure has the stronger than NBU property of being an increasing failure rate on average (IFRA) distribution. We do not know if this result can be extended to the case  $n > 2$ .

(iv) An interesting special case obtains when we take

$$\lambda_1(W) = \lambda_1 C \frac{a_1}{\sum_{j \in W} a_j}, \quad 1 \in W$$

where  $a_1$  are given nonnegative numbers. Such a situation would arise from the following weighted load sharing model: Suppose that an  $n$  component system is subject to a constant load pressure  $C$  which must be allocated among the working components. Suppose also that the allocation is determined by a set of weights  $a_1, \dots, a_n$  such that if at any time  $W$  is the set of working components then the load taken on by component  $i$ ,  $i \in W$ , is  $C a_i / \sum_{j \in W} a_j$ . If in addition we suppose that the failure rate of component  $i$  is proportional to (with proportionality constant  $\lambda_i$ ) the load, it is

assuming then the above obtains. For this model, it can be shown (see [5]) that for a parallel system the time until system failure (that is the time until all components have failed) is an increasing failure rate (IFR) random variable.

## 2. LAPLACE TRANSFORM OF SYSTEM LIFE

For a given structure one can, upon conditioning on the order in which the components fail, obtain an expression for the Laplace transform of  $T$ , the time of system failure. For instance, suppose a parallel system which fails when all components fail. Then letting

$$\bar{\lambda}_1(W) = \lambda_1(W^C), \text{ where } W^C = \text{complement of } W$$

we have

$$E[e^{-sT}] = \sum_{(i_1, \dots, i_n) \in \underline{P}} P(i_1, \dots, i_n) \prod_{k=1}^n \left[ \frac{\bar{\lambda}_{i_1}(i_1, \dots, i_{k-1})}{s + \sum_{j=k}^n \bar{\lambda}_{i_j}(i_1, \dots, i_{k-1})} \right]$$

where

$\underline{P}$  is the set of all  $n!$  permutations of  $1, 2, \dots, n$

and

$$P(i_1, i_2, \dots, i_n) = \frac{\bar{\lambda}_{i_1}}{\sum_{j=1}^n \bar{\lambda}_{i_j}} \frac{\bar{\lambda}_{i_2}(i_1)}{\sum_{j=2}^n \bar{\lambda}_{i_j}(i_1)} \dots \frac{\bar{\lambda}_{i_k}(i_1, \dots, i_{k-1})}{\sum_{j=k}^n \bar{\lambda}_{i_j}(i_1, \dots, i_{k-1})} \dots 1$$

is the probability that components fail in that order.

The above can easily be understood by noting that given that the components fail in order  $i_1, i_2, \dots, i_n$  the time between successive failure components are independent exponentials with rates

$$\sum_{j=1}^n \bar{\lambda}_{i_j}, \sum_{j=2}^n \bar{\lambda}_{i_j}(i_1), \dots, \bar{\lambda}_{i_n}(i_1, \dots, i_{n-1}).$$

### 3. SIMULATING THE PROCESS

Let  $T_i$  denote the failure time of component  $i$ . The random vector  $(T_1, \dots, T_n)$  can most easily be simulated as follows: Let  $X_i$ ,  $i = 1, \dots, n$  be independent exponentials with respective rates  $\bar{\lambda}_i$ ,  $i = 1, \dots, n$ . Now order the  $X_i$  and let  $i_1, i_2, \dots, i_n$  be such that

$$X_{i_1} < X_{i_2} < \dots < X_{i_n}.$$

Now set

$$T_{i_1} = X_{i_1}$$

$$T_{i_2} = X_{i_1} + (X_{i_2} - X_{i_1}) \frac{\sum_{j=2}^n \bar{\lambda}_{i_j}}{\sum_{j=2}^n \bar{\lambda}_{i_j}(i_1)}$$

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$$T_{i_k} = T_{i_{k-1}} + (X_{i_k} - X_{i_{k-1}}) \frac{\sum_{j=k}^n \bar{\lambda}_{i_j}}{\sum_{j=k}^n \bar{\lambda}_{i_j}(i_1, \dots, i_{k-1})}, \quad k = 2, \dots, n.$$

The above follows by first noting that given  $i_1, \dots, i_n$ , it follows from the lack of memory property of the exponential that

$$X_{i_1} \sum_{j=1}^n \bar{\lambda}_{i_j}, (X_{i_2} - X_{i_1}) \sum_{j=2}^n \bar{\lambda}_{i_j}, \dots, (X_{i_k} - X_{i_{k-1}}) \sum_{j=k}^n \bar{\lambda}_{i_j}, \dots$$

are independent exponentials with rates 1. The denominator term

$\sum_{j=k}^n \bar{\lambda}_{ij}(i_1, \dots, i_{k-1})$  in the definition of  $T_{i_k}$  thus gives the exponential its appropriate rate.

Remark:

When  $n = 2$  and  $\bar{\lambda}_1 \leq \bar{\lambda}_1(j)$ ,  $j \neq 1$ , the above expresses  $T_1, T_2$  as an increasing homogeneous function of  $X_1, X_2$ . This was noted and used in [6] to show that the  $T_i$  are associated (follows from the fact they are increasing functions of the independent random variables  $X_i$  (see [1] for a proof of this)) and also that system life is IFRA (follows from the fact that the function is not only increasing but also homogeneous--see [4] for a proof of this). Unfortunately, when  $n > 2$ , these functions are no longer increasing.

#### 4. THE MODEL WITH REPAIR

Let us suppose that failed components are repaired. Specifically, suppose that the repair rate of component  $i$  when the set of working components is  $W$ ,  $i \notin W$ , is  $\mu_i(W)$ . This gives rise to a continuous time Markov chain with  $2^n$  states--all possible subsets  $W \subseteq \{1, 2, \dots, n\}$ . To solve for the steady state probabilities is in general a difficult task but it simplifies in the following special case:

##### Special case:

For functions  $f(k)$ ,  $g(k)$ ,  $k = 0, 1, \dots, n$  and positive constants  $\lambda_k$ ,  $\mu_k$ ,  $k = 1, \dots, n$ ,

$$\lambda_j(W) = \lambda_j f(|W|) \quad , \quad j \in W$$

$$\mu_i(W) = \mu_i g(|W|) \quad , \quad i \notin W .$$

##### Proposition 2:

Under the conditions of the above special case, the stationary probabilities of the set of working components is given as follows: For  $W = \{i_1, \dots, i_k\}$ ,

$$P(\{i_1, \dots, i_k\}) = \frac{\mu_{i_1} \mu_{i_2} \dots \mu_{i_k} g(k-1) \dots g(0)}{\lambda_{i_1} \lambda_{i_2} \dots \lambda_{i_k} f(k) \dots f(1)} P(\phi)$$

where  $P(\phi)$  is the stationary probability that all components are failed and can be obtained by summing the above over all  $W$  and equating to 1. In addition the chain, in steady state, is time reversible.



Proof:

To verify the above, all we need check is that the proposed stationary probabilities satisfy the time reversibility equations. That is we need check that, for the proposed stationary probabilities,

$$P(\{i_1, \dots, i_k\}) \lambda_{i_1} f(k) = P(\{i_2, \dots, i_k\}) \mu_{i_1} g(k-1) .$$

But this is immediate and so the result follows. ||

## REFERENCES

- [1] Barlow, R. E. and F. Proschan, STATISTICAL THEORY OF RELIABILITY AND LIFE TESTING, Holt, Rinehart and Winston, Inc., pp. 1-14, (1975).
- [2] Freund, J. E., "A Bivariate Extension of the Exponential Distribution," Journal of the American Statistical Association, Vol. 56, pp. 971-977, (1961).
- [3] Friday, D. S. and G. P. Patil, "A Bivariate Exponential Model with Applications to Reliability and Computer Generation of Random Variables," THEORY AND APPLICATIONS OF RELIABILITY, Vol. I, C. P. Tsokos and I. N. Shimi, eds., pp. 527-549, (1977).
- [4] Ross, S. M., "Multivalued State Component Systems," The Annals of Probability, Vol. 7, No. 2, pp. 379-383, (1979).
- [5] Schechner, Z., "Load Sharing Models and Their Life Distributions," ORC 80-19, Operations Research Center, University of California, Berkeley, (1980).
- [6] Shaked, M., "Extensions of the Freund Distribution with Applications in Reliability Theory," unpublished paper, (1982).

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